STOCHASTIC INSTABILITY OF MULTIDIMENSIONAL ANHARMONIC LATTICES

E. M. Krushkal'

UDC 531.011

A criterion is obtained for the stochastic instability of multidimensional anharmonic lattices.

Consider a medium described by the nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\left[1 + \lambda\left(\frac{\partial u}{\partial x}\right)^2\right] + \mu\frac{\partial^2 u}{\partial y^2}\left[1 + \lambda\left(\frac{\partial u}{\partial y}\right)^2\right] + \gamma\frac{\partial^2 u}{\partial z^2}\left[1 + \lambda\left(\frac{\partial u}{\partial z}\right)^2\right] \tag{0.1}$$

in which $u \equiv u(t, x, y, z)$, x, y, z are the spatial coordinates, t is the time, and λ is the coefficient of nonlinearity.

The anharmonic lattice model corresponding to Eq. (0.1) has the form

$$\begin{aligned}\frac{d^2 u_{l,m,p}}{dt^2} = {} & u_{l+1,m,p} - 2u_{l,m,p} + u_{l-1,m,p} + \beta\left[(u_{l+1,m,p} - u_{l,m,p})^3 - (u_{l,m,p} - u_{l-1,m,p})^3\right] + \\ & + \mu(u_{l,m+1,p} - 2u_{l,m,p} + u_{l,m-1,p}) + \beta\mu\left[(u_{l,m+1,p} - u_{l,m,p})^3 - (u_{l,m,p} - u_{l,m-1,p})^3\right] + \\ & + \gamma(u_{l,m,p+1} - 2u_{l,m,p} + u_{l,m,p-1}) + \beta\gamma\left[(u_{l,m,p+1} - u_{l,m,p})^3 - (u_{l,m,p} - u_{l,m,p-1})^3\right]\end{aligned} \tag{0.2}$$

where $\beta = \lambda/3$.

For values of the coefficients

$$\mu = \gamma = 0$$

Eq. (0.1) describes a nonlinear string, to the analysis of which has been devoted a large number of papers in connection with the well-known problem of the buildup of thermodynamic equilibrium and ergodicity in a system of nonlinearly coupled oscillators (see, e.g. [1-3] and the bibliographies included therein).

Izrailev and Chirikov [2] have obtained an estimate of the stochastic instability limits, which separates the domain of quasi-periodic motion and the stochasticity domain for an array of nonlinearly coupled harmonic oscillators in the one-dimensional case.

In the present article we generalize this result to multidimensional (two- and three-dimensional) lattices with rigidly clamped boundaries.

1. Two-Dimensional Lattice

In Eq. (0.2) we put $\gamma = 0$ (the third subscript can be dropped). We let N_1 and N_2 be the numbers of oscillators in the respective x and y directions.

We transform to normal coordinates:

$$u_{l,m} = \frac{2}{\sqrt{N_1 N_2}} \sum_{k=1}^{N_1-1} \sum_{r=1}^{N_2-1} Q_{kr} \sin\frac{\pi lk}{N_1} \sin\frac{\pi mr}{N_2} \tag{1.1}$$

For a slowly varying quantity

$$Q_{kr}(t) = c_{kr}(t)\cos\theta_{kr}(t) \tag{1.2}$$

where c_{kr} and θ_{kr} are the normal-mode amplitude and phase, we obtain the following equation from (0.2):

Novosibirsk. Translated from Zhurnal Prikladnoi Mekahniki i Tekhnicheskoi Fiziki, No. 6, pp. 173-178, November-December, 1972. Original article submitted December 2, 1971.

$$\frac{d^2Q_{kr}}{dt^2} + \omega_{kr}^2 Q_{kr}(1 - AQ_{kr}^2) = \frac{\beta}{16N_1N_2}\sum_{m,n} F_{km,rn}\cos\theta_{km,rn} \tag{1.3}$$

$(k = 1, 2, \ldots, N_1 - 1, r = 1, 2, \ldots, N_2 - 1)$ where

$$A \approx \frac{3\beta}{8N_1N_2\omega_{kr}^2}[\omega_{k0}^4(2 - \omega_{k0}^2) + \mu\omega_{0r}^4(2 - \omega_{0r}^2)]$$

$$\omega_{kr} = 2\left(\sin^2\frac{\pi k}{2N_1} + \mu\sin^2\frac{\pi r}{2N_2}\right)^{1/2}$$

$$\omega_{k0} = 2\sin\frac{\pi k}{2N_1}, \quad \omega_{0r} = 2\sin\frac{\pi r}{2N_2}$$

where $F_{km,rn}$ represents the amplitudes of the external forces acting on the oscillator of frequency $\omega_{km,nr}(t)$. We choose not to write out the cumbersome explicit form of the right-hand side of Eq. (1.3), analyzing instead a number of limiting cases.

The stochasticity criterion, which is based on the condition of overlapping of resonances [4], has the form

$$\varkappa \sim |\psi_{km,rn}|(\Delta\omega)^{-1} \sim 1 \tag{1.4}$$

where

$$|\dot\psi_{km,rn}| \approx \sqrt{\frac{\beta F_{km,rn}}{8N_1N_2\omega_{kr}}\frac{d\Omega_{km,rn}}{dc_{kr}}} \tag{1.5}$$

characterizes the dimension of the separatrix on the phase plane

$$\Omega_{km,rn} = \omega_{km,rn}(t) - \omega_{kr}(t)$$

and $\Delta\omega$ is the separation of the resonances.

Excitation of Low Modes ($k \ll N_1$, $r \ll N_2$). In this case the normal mode frequencies are

$$\omega_{kr} \approx \pi\left[\frac{k^2}{N_1^2} + \mu\frac{r^2}{N_2^2}\right]^{1/2} \tag{1.6}$$

Let k and r be the middle order numbers of the excited modes in the respective intervals Δk and Δr, and let N_b be the number of excited modes:

$$N_b \sim \Delta k\Delta r \tag{1.7}$$

We calculate the number of nondegenerate resonance relations N_p on the right-hand side of (1.3) of the type

$$\sum_{i=1}^{4} n_i\omega_i = 0 \tag{1.8}$$

where the n_i are relative primes [we recall that the cubic nonlinear term in (0.2) gives rise to fourfold interaction]. It is important here to allow for the fact that in the given limit of oscillators stationed along one straight line through the origin linear relations are obtained with respect to their order numbers according to (1.6) and (1.8), as in the one-dimensional case. Therefore, the number of independent resonance relations (1.8) decreases due to degeneracy in comparison with the number of possible resonances in the system.

We have as a result

$$N_p = 8C_{N_b-1}^3 - N_0 + N_l \tag{1.9}$$

where C_n^m is the number of combinations of n elements m at a time and the following notation is introduced:

$$N_0 = 8C_{a-1}^3, \quad N_l = a/2 \quad (a = \min(\Delta k, \Delta r)) \tag{1.10}$$

The factor 1/2 in the expression for N_l in (1.10) accounts for the presence of symmetric interaction associated with cubic nonlinearity in the degenerate case, as in the one-dimensional case. We also note that with allowance for this fact the degeneracy effect only has to be included if $a \geq 8$, because this is the only case in which the corresponding frequency combination in (1.8) is possible.

We now estimate the magnitude of the average force $\langle F\rangle$ acting on the oscillator of order kr on the nearest resonance side. According to (1.3) and (1.9) and with allowance for the random phase condition we have

$$\langle F\rangle \sim \beta \frac{F_{km,rn}}{16N_1N_2} \sim \frac{\pi^4 (8C^3_{N_b-1})^{1/2}\beta \langle c^3\rangle}{N_1N_2(N_p)^{1/2}} \left(\frac{k^4}{N_1^4} + \mu \frac{r^4}{N_2^4}\right) \quad (1.11)$$

where $\langle c^3\rangle$ is the average combination of amplitudes in the vicinity of the excited modes. The correction to the frequency ω_{kr} for nonlinearity is

$$\Omega_{km,rn} \sim 9\left[32N_1N_2\left(\frac{k^2}{N_1^2} + \mu\frac{r^2}{N_2^2}\right)^{1/2}\right]^{-1} \beta\left(\frac{k^4}{N_1^4} + \mu\frac{r^4}{N_2^4}\right)\langle c^2\rangle \quad (1.12)$$

The average spacing between resonances can be estimated as the intervals between the frequency vectors in the vicinity of the excited modes:

$$|\omega_{k-\Delta k/2,\, r-\Delta r/2}| \leqslant |\omega_{kr}| \leqslant |\omega_{k+\Delta k/2,\, r+\Delta r/2}|$$

occupied by N_p resonances, using (1.6) and (1.9):

$$\Delta\omega \sim \frac{\pi}{N_p}\left[\left(\frac{k\Delta k}{N_1^2}\right)^2 + \mu\left(\frac{r\Delta r}{N_2^2}\right)^2\right]^{1/2}\left(\frac{k^2}{N_1^2} + \mu\frac{r^2}{N_2^2}\right)^{-1/2} \quad (1.13)$$

Taking (1.2) into account, we obtain the following maximal estimate from (1.1):

$$u^2_{\max} = 4(\langle u\rangle)^2 \sim 2\langle c^2\rangle \Delta k\Delta r (N_1N_2)^{-1} \quad (1.14)$$

We introduce the notation

$$\Phi_2 \equiv \left(\frac{\partial u}{\partial x}\right)^2 + \mu\left(\frac{\partial u}{\partial y}\right)^2 = \pi^2\left(\frac{k^2}{N_1^2} + \mu\frac{r^2}{N_2^2}\right)u^2_{\max} \quad (1.15)$$

$$\sigma_1 = \frac{rN_1}{kN_2}, \quad \Delta\sigma_1 = \frac{r\Delta rN_1^2}{k\Delta kN_2^2} \quad (1.16)$$

Determining the quantity β_* from condition (1.4) with regard for (1.5) and (1.11)-(1.13), we obtain the following estimate for the stochasticity limit:

$$\beta_*\Phi_2 \sim \frac{2(1+\mu\sigma_1^2)[1+\mu(\Delta\sigma_1)^2]^{1/2}(\Delta k)^2\Delta r}{k(1+\mu\sigma_1^4)[8C^3_{N_b-1}(N_p)^3]^{1/4}} \quad (1.17)$$

For $\Delta k, \Delta r \gg 1$ we have

$$8C^3_{N_b-1} \sim (\Delta k)^3(\Delta r)^3, \quad 8C^3_{N_b-1} \gg N_0 \gg N_l$$

and (1.17) assumes the form

$$\beta_*\Phi_2 \sim \frac{4(1+\mu\sigma_1^2)(1+\mu\Delta\sigma_1^2)^{1/2}}{3k(1+\mu\sigma_1^4)\Delta k(\Delta r)^2}$$

As $\mu \to 0$ the domain of the excited modes shrinks to a line on the x axis:

$$8C^3_{N_b-1} \sim (\Delta k)^3, \quad N_0 \sim (\Delta k)^3, \quad N_l \sim \Delta k/2$$
$$\Phi_2 \to \Phi_1 \equiv (\partial u/\partial x)^2$$

so that

$$\beta_*\Phi_1 \sim k^{-1}(\Delta k)^{1/2}$$

which corresponds to the estimate obtained in [2] for the one-dimensional case.

Excitation of High Modes ($k \approx N_1$, $r \approx N_2$). Now the resonance frequencies are equal to

$$\omega_{km,rn} \approx 2\left[\cos^2\frac{\pi\sqrt{(k-N_1)^2+2m}}{2N_1} + \mu\cos^2\frac{\pi\sqrt{(r-N_2)^2+2n}}{2N_2}\right]$$
$$(m, n = 0, \pm 1, \pm 2\ldots)$$

The average spacing between closest resonances in the first approximation is

$$\Delta\omega_{\pm} = \omega_{k,\, m\pm1,\, r,\, n\mp1} - \omega_{km,\, rn} \approx \frac{\pi^2}{2(1+\mu)^{1/2}} \left| \frac{1}{N_1^2} \pm \frac{\mu}{N_2^2} \right| \tag{1.18}$$

The next-higher-order correction gives the distance between the fine-structure resonances:

$$\Delta\omega_1 \sim (\pi / N)^4 \quad \text{for} \quad N_1 = N_2 = N$$

but, as is readily noted, in the given limit ($\Delta k \ll N_1$, $\Delta r \ll N_2$) complete overlapping of this system of resonances does not take place, so that for $N_1 \approx N_2 \approx N$ and $\mu \approx 1$ it is required to choose the upper sign in (1.18) due to the addditional degeneracy.

With regard for (1.3), (1.4), and (1.14)-(1.16) we obtain the following estimate of the stochasticity limit (for $\Delta\omega_- \gg \Delta\omega_1$):

$$\beta_* \Phi_2 \sim \frac{2\pi^2 k^2 (1+\mu\sigma_1^2)\, \Delta k \Delta r}{(1+\mu)^{1/2} N_1^2} \left| \frac{1}{N_1^2} - \mu \frac{1}{N_2^2} \right| \tag{1.19}$$

For $N_1 \sim N_2 \sim N$ and $\mu \sim 1$ we have

$$\Delta\omega_+ \gg \Delta\omega_1 \sim \Delta\omega_- \tag{1.20}$$

$$\beta_* \Phi_2 \sim 8/3\pi^2 \ \Delta k \Delta r \ (k^2 + r^2) / N^4$$

As $\mu \to 0$ (one-dimensional case) (1.19) and (1.20) go over to

$$\beta_* \Phi_1 \sim \pi^2 k^2 \Delta k / N^4$$

consistent with the result of [2].

Mixed Case ($k \approx N_1$, $r \ll N_2$). Suppose that the highest modes are excited in one direction and the lowest in the other direction. The resonance frequencies are

$$\omega_{km,\, rn} \approx 2 \left[\cos^2 \frac{\pi \sqrt{(k-N_1)^2 + 2m}}{2N_1} + \mu \sin^2 \frac{\pi \sqrt{r^2 + 2n}}{2N_2} \right]^{1/2}$$
$$(m, n = 0, \pm1, \pm2 \ldots)$$

Then the estimate of the stochasticity limit coincides with expressions (1.19) and (1.20), i.e., is determined by the presence of the highest modes.

2. Three-Dimensional Lattice

Using the results of the preceding section, we readily obtain estimates for the three-dimensional case.

The expansion in normal modes now has the form

$$u_{l,\, m,\, p} = \frac{2\sqrt{2}}{\sqrt{N_1 N_2 N_3}} \sum_{k=1}^{N_1-1} \sum_{r=1}^{N_2-1} \sum_{s=1}^{N_3-1} Q_{krs} \sin \frac{\pi l k}{N_1} \sin \frac{\pi m r}{N_2} \sin \frac{\pi p s}{N_3}$$

where N_1, N_2, N_3 are the numbers of oscillators of the lattice in the respective x, y, and z directions.

The numbers of excited modes in the intervals Δk, Δr, and Δs are

$$N_b \sim \Delta k \Delta r \Delta s \tag{2.1}$$

The normal-mode frequencies are

$$\omega_{krs} = 2 \left(\sin^2 \frac{\pi k}{2N_1} + \mu \sin^2 \frac{\pi r}{2N_2} + \gamma \sin^2 \frac{\pi s}{2N_3} \right)^{1/2}$$
$$(k = 1, 2, \ldots, N_1 - 1;\ r = 1, 2, \ldots, N_2 - 1;\ s = 1, 2, \ldots, N_3 - 1)$$

For N_p we can once again invoke expression (1.9) with regard for the fact that now

$$a = \min(\Delta k, \Delta r, \Delta s)$$

and the domain of the excited modes is given by (2.1).

We introduce the notation

$$\Phi_3 \equiv \left(\frac{\partial u}{\partial x} \right)^2 + \mu \left(\frac{\partial u}{\partial y} \right)^2 + \gamma \left(\frac{\partial u}{\partial z} \right)^2 = \pi^2 \left(\frac{k^2}{N_1^2} + \mu \frac{r^2}{N_2^2} + \gamma \frac{s^2}{N_3^2} \right) u_{\max}^2$$

where

$$u_{\max}^2 \sim \frac{2\langle c^2\rangle N_b}{N_1 N_2 N_3}, \quad \sigma_1 = \frac{rN_1}{kN_2}, \quad \sigma_2 = \frac{sN_1}{kN_3}$$

$$\Delta\sigma_1 = \frac{r\Delta r N_1^2}{k\Delta k N_2^2}, \quad \Delta\sigma_2 = \frac{s\Delta s N_1^2}{k\Delta k N_3^2}$$

Then the estimates of the stochasticity limits are obtained in the following form:

Low modes ($k \ll N_1$, $r \ll N_2$, $s \ll N_3$):

$$\beta_* \Phi_3 \sim \frac{4(1+\mu\sigma_1^2+\gamma\sigma_2^2)\,[1+\mu(\Delta\sigma_1)^2+\gamma(\Delta\sigma_2)^2]^{1/2}(\Delta k)^2\Delta r\Delta s}{k(1+\mu\sigma_1^4+\gamma\sigma_2^4)\,[8C^3_{N_b-1}(N_p)^3]^{1/4}} \tag{2.2}$$

and for $\Delta k, \Delta r, \Delta s \gg 1$ relation (2.2) assumes the form

$$\beta_* \Phi_3 \sim \frac{3(1+\mu\sigma_1^2+\gamma\sigma_2^2)\,[1+\mu(\Delta\sigma_1)^2+\gamma(\Delta\sigma_2)^2]^{1/2}}{k(1+\mu\sigma_1^4+\gamma\sigma_2^4)\,\Delta k\,(\Delta r)^2(\Delta s)^2}.$$

High modes ($k \approx N_1$, $r \approx N_2$, $s \approx N_3$):

$$\beta_* \Phi_3 \sim 4\pi^2 \frac{k^2(1+\mu\sigma_1^2+\gamma\sigma_2^2)\,\Delta k\Delta r\Delta s}{N_1^2(1+\mu+\gamma)^{1/2}} \left| \frac{1}{N_1^2} \pm \frac{\mu}{N_2^2} \mp \frac{\gamma}{N_3^2} \right| \tag{2.3}$$

The signs in (2.3) are chosen so as to minimize the expression, thus corresponding to the shortest spacing between resonances. Estimate (2.3) cannot be used for $N_1 \approx N_2 \approx N$, $\gamma \approx 1-\mu$. For this case, taking into account the discussion to Eq. (1.18), we have

$$\beta_* \Phi_3 \sim \frac{17}{3} \frac{\pi^2 \Delta k \Delta r \Delta s}{N^4}(k^2+r^2+s^2) \tag{2.4}$$

In a mixed situation (say, $k \approx N_1$, $r \ll N_2$, $s \ll N_3$) we obtain the same estimates (2.3)-(2.4).

As the foregoing estimates imply, the growth of stochasticity is facilitated with increasing degrees of freedom. For a reliable verification of the analytical estimates it would be desirable to conduct experiments with a large number of oscillators.

The authors are indebted to B. V. Chirikov for a discussion.

LITERATURE CITED

1. E. Fermi, J. Pasta, and S. Ulam, Studies of Nonlinear Problems, Los Alamos Rep. LA-1940 (1955).
2. F. M. Izrailev and B. V. Chirikov, "Statistical properties of a nonlinear string," Dokl. Akad. Nauk SSSR, 166, No. 1 (1966).
3. N. Saito, N. Ooyama, Y. Aizawa, and H. Hirooka, "Computer experiments on ergodic problems in anharmonic lattice vibrations," Prog. Theor. Phys. Suppl. (Japan), No. 45, 209-230 (1970).
4. B. V. Chirikov, Studies in the Theory of Nonlinear Resonance and Stochasticity, Doctoral Dissertation Nuclear Physics Institute, Siberian Branch, Acad. Sciences USSR, Novosibirsk (1969).